

CSC 321: Data Structures

Fall 2013

Counting and problem solving

- mappings, bijection rule
- sequences, product rule, sum rule
- generalized product rule, permutations
- division rule, inclusion/exclusion
- pigeonhole principle

1

Bijections

it is often easier to count one thing by counting another

e.g., to count sheep, count legs and divide by 4

recall:

a *function* ($f : D \rightarrow R$) is a mapping from elements of a domain D to elements of a range R

e.g., $\text{sqrt} : \mathbb{R} \rightarrow \mathbb{R}$ $\text{abs} : \mathbb{R} \rightarrow \mathbb{R}^+$

a *bijective function* or *bijection* is a one-to-one mapping between two sets

e.g., current Creighton students & staff, active netIDs

each student/staff has a unique netID; each netID has a unique student/staff

e.g., $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = 2x+1$

each x maps to a unique $f(x)$; each $f(x)$ has a unique x

Bijection Rule: If there is a bijection $f : A \rightarrow B$, then $|A| = |B|$.

2

Application: bijection rule

A = # of ways to assign letter grades (A/B/C/D/F) to 10 students

B = # of 14 bit patterns with exactly four 1's

consider grades from A:

2 A's, 4 B's, 1 C, 2 D's, 1 F

map to a bit pattern from B:

00100001010010

this is a 1-to-1 mapping, so $|A| = |B|$

- if we can determine the size of either set, then we know the size of the other
- $|B| = \binom{14}{4} = 14!/10!4! = 14*13*12*11/4*3*2*1 = 7*13*11 = 1,001$

3

Sequences

general strategy:

focus on techniques for counting sequences, then
for each counting problem to be solved, try to map it into sequences

recall:

if P_1, P_2, \dots, P_n are sets, then $P_1 \times P_2 \times \dots \times P_n$ is the set of all sequences where
the 1st term is from P_1 , 2nd term is from P_2 , ..., nth term is from P_n

- e.g., $C = \{\text{red, blue}\}$, $N = \{1, 2, 3\}$,
 $C \times N = \{\text{red-1, red-2, red-3, blue-1, blue-2, blue-3}\}$

Product Rule: if P_1, P_2, \dots, P_n are sets, then

$$|P_1 \times P_2 \times \dots \times P_n| = |P_1| * |P_2| * \dots * |P_n|$$

4

Application: product rule

suppose we are trying to build a computer out of components

$$P = \{ i5, i7 \}$$

$$R = \{ 2GB, 4GB, 8GB \}$$

$$C = \{ 1MB, 2MB, 4MB \}$$

how many combinations of processor, RAM & cache are there to choose from?

$$\bullet |P \times R \times C| = |P| * |R| * |C| = 2 * 3 * 3 = 18$$

{ i5-2GB-1MB, i5-2GB-2MB, i5-2GB-4MB,
i5-4GB-1MB, i5-4GB-2MB, i5-4GB-4MB,
i5-8GB-1MB, i5-8GB-2MB, i5-8GB-4MB,
i7-2GB-1MB, i7-2GB-2MB, i7-2GB-4MB,
i7-4GB-1MB, i7-4GB-2MB, i7-4GB-4MB,
i7-8GB-1MB, i7-8GB-2MB, i7-8GB-4MB }

5

Application: product rule

how many possible subsets of a set of N elements?

$$S = \{ x_1, x_2, \dots, x_n \}$$

can map each subset into a sequence of N bits: $b_i = 1 \rightarrow x_i$ in subset

$$\{ x_1, x_4, x_5 \} \leftrightarrow 10011000\dots 0$$

- for an N-element set, count number of N-bit sequences

$$|\{0,1\} \times \{0,1\} \times \{0,1\} \times \dots \times \{0,1\}| = |\{0,1\}|^n = 2^n$$

e.g., $S = \{ a, b, c \}$

$$\text{subsets of } S = \{ \{ \}, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, b, c \} \}$$

$$|\text{subsets of } S| = 2^{|S|} = 2^3 = 8$$

6

Combining rules

many problems involve a combination of counting methods

Sum Rule: if S_1, \dots, S_n are disjoint sets, then $|S_1 \cup \dots \cup S_n| = |S_1| + \dots + |S_n|$

Division Rule: if $f: A \rightarrow B$ is k-to-1, then $|A| = k * |B|$.

suppose a computer system requires 6-8 character passwords, consisting of letters & digits that must start with a letter

- $\text{alpha} = \{ a, b, \dots, z, A, B, \dots, Z \}$
- $\text{anum} = \text{alpha} \cup \{ 0, 1, 2, \dots, 9 \}$

$$\begin{aligned} \text{passwords} &= (\text{6-char passwords}) \cup (\text{7-char passwords}) \cup (\text{8-char passwords}) \\ &= (\text{alpha} \times \text{anum}^5) \cup (\text{alpha} \times \text{anum}^6) \cup (\text{alpha} \times \text{anum}^7) \end{aligned}$$

$$\begin{aligned} |\text{passwords}| &= |(\text{alpha} \times \text{anum}^5) \cup (\text{alpha} \times \text{anum}^6) \cup (\text{alpha} \times \text{anum}^7)| \\ &= |\text{alpha} \times \text{anum}^5| + |\text{alpha} \times \text{anum}^6| + |\text{alpha} \times \text{anum}^7| \quad [\text{Sum Rule}] \\ &= |\text{alpha}| * |\text{anum}|^5 + |\text{alpha}| * |\text{anum}|^6 + |\text{alpha}| * |\text{anum}|^7 \quad [\text{Product Rule}] \\ &= 52 * 62^5 + 52 * 62^6 + 52 * 62^7 \\ &= 186,125,210,680,448 \end{aligned}$$

7

Generalized product rule

the product rule assumes that the choices are independent

- pick-4 lotto: 4 bins of balls numbered 0-9.
 $|\{0..9\} \times \{0..9\} \times \{0..9\} \times \{0..9\}| = |\{0..9\}|^4 = 10^4 = 1,000$

what about a lottery where balls numbered 1-42 are in a bin & draw 4?

Generalized Product Rule: Let S be a set of length-k sequences. If there are:

- n_1 possible first entries,
- n_2 possible second entries for each first entry,
- n_3 possible third entries for each combination of first and second entries, etc.

then:

$$|S| = n_1 * n_2 * n_3 * \dots * n_k$$

- |4-ball draws| = $42 * 41 * 40 * 39 = 2,686,320$

8

Counting subsets

the lottery problem is an example of a more generic problem

- how many k-element subsets of an n-element set are there?
- e.g., how many 4-ball lottery numbers are there (assuming 42 balls)?
- e.g., how many 5-card poker hands are there (assuming 52 cards)?
- e.g., how many 3-topping pizzas are there (assuming 10 toppings)?

"n choose k" is such a common expression that it has its own notation: $\binom{n}{k}$

- can map any permutation of n items into a k-element subset by simply taking the first k elements of in the permutation
- this is not a 1-1 mapping though, since any arrangement of the k-element prefix and (n-k)-element suffix yields the same subset
- Division rule $\rightarrow |\text{perms}| = |\text{perms of prefix}| * |\text{perms of suffix}| * \binom{n}{k}$

$$\rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

9

Pascal's identity

"n choose k" can also be defined using recursion

- select a "special" element of the list (e.g., lottery ball #1)

$|\text{subsets}| = |\text{subsets that include special element}| + |\text{subsets that don't}|$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

e.g., choose 4 out of 42 lottery balls =
choose 3 out of 41 balls (includes 1) +
choose 4 out of 41 balls (doesn't include 1)

```
def choose1(n, k):
    return math.factorial(n)/(math.factorial(k)*math.factorial(n-k))

def choose2(n, k):
    if k == n:
        return 1
    elif k == 1:
        return n
    else:
        return choose2(n-1, k-1) + choose2(n-1, k)
```

10

Application: hand counting

how many different 4-of-a-kind hands are there?

- can define a bijection with the sequence:
rank of matching cards, rank of 5th card, suit of 5th card

- Generalized Product Rule:

$$\begin{aligned} |\text{sequences}| &= |\text{rank of matching}| * |\text{rank of 5}^{\text{th}}| * |\text{suit of 5}^{\text{th}}| \\ &= 13 * 12 * 4 \\ &= 624 \end{aligned}$$

- there are "52 choose 5" possible hands = $52! / (47!5!) = 2,598,960$
- odds of drawing 4-of-a-kind are $624 / 2598960 \approx 1/4165$

11

Application: hand counting

how many different full house hands are there?

- can define a bijection with the sequence:
rank of 3, suits of 3, rank of 2, suits of 2

- Generalized Product Rule:

$$\begin{aligned} |\text{sequences}| &= |\text{rank of 3}| * |\text{suits of 3}| * |\text{rank of 2}| * |\text{suits of 2}| \\ &= 13 * (4 \text{ choose } 3) * 12 * (4 \text{ choose } 2) \\ &= 13 * 4! / (3!1!) * 12 * 4! / (2!2!) \\ &= 13 * 24/6 * 12 * 24/4 \\ &= 3,744 \end{aligned}$$

- odds of drawing a full house are $3744 / 2598960 \approx 1/694$

12

Application: hand counting

how many different 2-pair hands are there?

- can define a mapping with the sequence:
rank of 1st pair, suits of 1st pair, rank of 2nd pair, suits of 2nd pair,
rank of extra, suit of extra

- Generalized Product Rule:

$$\begin{aligned} |\text{sequences}| &= 13 * (4 \text{ choose } 2) * 12 * (4 \text{ choose } 2) * 11 * 4 \\ &= 13 * 4!/(2!2!) * 12 * 4!/(2!2!) * 11 * 4 \\ &= 13 * 24/4 * 12 * 24/4 * 11 * 4 \\ &= 247,104 \end{aligned}$$

- odds of drawing 2-pairs are $247104/2598960 \approx 1/11$

WRONG: the mapping is not a bijection
3C3SQHQ SAD and QHQ S3C3SAD map to same sequence
2-to-1 mapping, so $247104/2 = 123,552$ hands ($\approx 1/22$)

13

Set overlap

recall the Sum Rule: if S_1, \dots, S_n are disjoint sets, then $|S_1 \cup \dots \cup S_n| = |S_1| + \dots + |S_n|$

- but what if the sets are not disjoint?

suppose JM&C has 50 ICS majors and 15 DDD majors

- that doesn't necessarily mean 65 distinct majors
total majors = ICS majors + DDD majors – (dual majors)

Inclusion/Exclusion Rule:

$$\begin{aligned} |S_1 \cup S_2 \cup \dots \cup S_n| = & \text{the sum of the sizes of the individual sets} - \\ & \text{the sizes of all two-way intersections} + \\ & \text{the sizes of all three-way intersections} - \\ & \text{the sizes of all four-way intersections} + \\ & \dots \end{aligned}$$

14

Pigeonhole principle

suppose your sock drawer contains black, brown, and white socks

- if you grab socks at random, how many must you grab to ensure a match?

Pigeonhole Principle: if $|X| > |Y|$, then for every total function $f : X \rightarrow Y$, there exist two different elements of X that are mapped to the same element of Y

- need to grab 4 socks to make $|\text{socks}| > |\text{colors}|$

- how many people must be in a room to ensure at least one shared birthday?
(only requires 57 for a 99% probability)