CSC 421: Algorithm Design Analysis

Spring 2014

Transform & conquer

- transform-and-conquer approach
- presorting
- balanced search trees, heaps
- Horner's Rule
- problem reduction

Transform & conquer

the idea behind transform-and-conquer is to transform the given problem into a slightly different problem that suffices

e.g., presorting data in a list can simplify many algorithms

- suppose we want to determine if a list contains any duplicates

  BRUTE FORCE: compare each item with every other item

  $(N-1) + (N-2) + \ldots + 1 = (N-1)N/2 \rightarrow O(N^2)$

  TRANSFORM & CONQUER: first sort the list, then make a single pass through the list and check for adjacent duplicates

  $O(N \log N) + O(N) \rightarrow O(N \log N)$

- finding the mode of a list, finding closest points, …

Balanced search trees

recall binary search trees – we need to keep the tree balanced to ensure O(N log N) search/add/remove

- OR DO WE?
- it suffices to ensure O(log N) height, not necessarily minimal height

transform the problem of "tree balance" to "relative tree balance"

several specialized structures/algorithms exist:

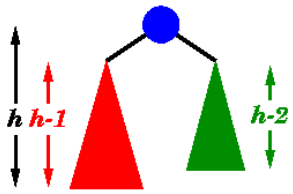
- AVL trees
- 2-3 trees
- red-black trees

3

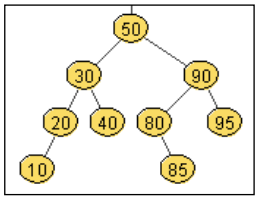
AVL trees

an AVL tree is a binary search tree where

- for every node, the heights of the left and right subtrees differ by at most 1



- first self-balancing binary search tree variant
- named after Adelson-Velskii & Landis (1962)



AVL tree

AVL property is weaker than perfect balance, but sufficient

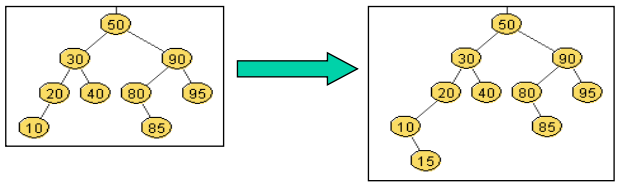
height of AVL tree with N nodes < 2 log(N+2)

→ searching is O(log N)

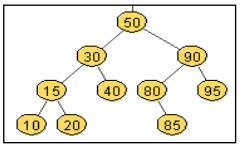
4

Inserting/removing from AVL tree

when you insert or remove from an AVL tree, imbalances can occur



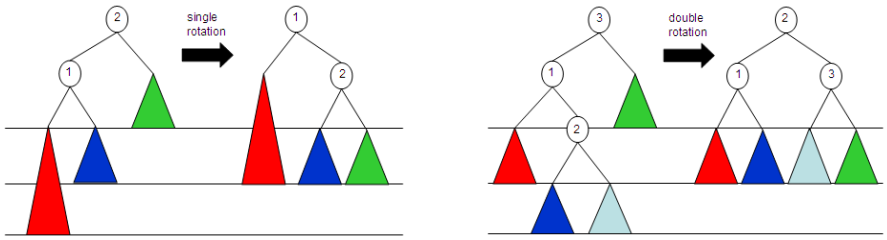
- if an imbalance occurs, must rotate subtrees to retain the AVL property



5

AVL tree rotations

there are two possible types of rotations, depending upon the imbalance caused by the insertion/removal



worst case, inserting/removing requires traversing the path back to the root and rotating at each level

- each rotation is a constant amount of work ➔ inserting/removing is O(log N)

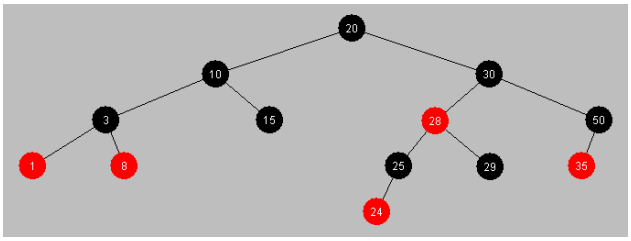
6

Red-black trees

a red-black tree is a binary search tree in which each node is assigned a color (either red or black) such that

1. *the root is black*
2. *a red node never has a red child*
3. *every path from root to leaf has the same number of black nodes*

- add & remove preserve these properties (complex, but still O(log N))
- red-black properties ensure that tree height < 2 log(N+1) → O(log N) search



7

TreeSets & TreeMaps

`java.util.TreeSet` uses *red-black trees* to store values

➔ O(log N) efficiency on add, remove, contains

`java.util.TreeMap` uses *red-black trees* to store the key-value pairs

➔ O(log N) efficiency on put, get, containsKey

thus, the original goal of an efficient tree structure is met

- even though the subgoal of balancing a tree was transformed into "relatively balancing" a tree

8

Scheduling & priority queues

many real-world applications involve optimal scheduling

- balancing transmission of multiple signals over limited bandwidth
- selecting a job from a printer queue
- selecting the next disk sector to access from among a backlog
- multiprogramming/multitasking

a *priority queue* encapsulates these three optimal scheduling operations:

- ✓ *add item (with a given priority)*
- ✓ *find highest priority item*
- ✓ *remove highest priority item*

- can be implemented as an unordered list
  → add is O(1), findHighest is O(N), removeHisghest is O(N)

- can be implemented as an ordered list
  → add is O(N), findHighest is O(1), removeHighest is O(1)

9

Heaps

Java provides a `java.util.PriorityQueue` class

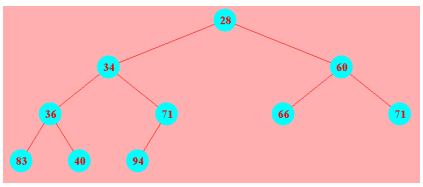
- the underlying data structure is not a list or queue at all
- it is a tree structure called a *heap*

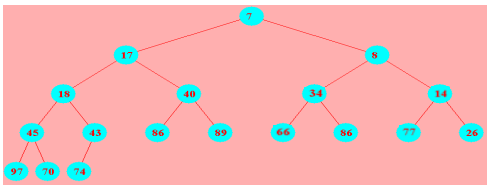
a *complete tree* is a tree in which

- all leaves are on the same level or else on 2 adjacent levels
- all leaves at the lowest level are as far left as possible
- note: a complete tree with N nodes will have minimal height = $\lfloor \log_2 N \rfloor + 1$

a *heap* is complete binary tree in which

- for every node, the value stored is ≤ the values stored in both subtrees
  (technically, this is a min-heap -- can also define a max-heap where the value is ≥)



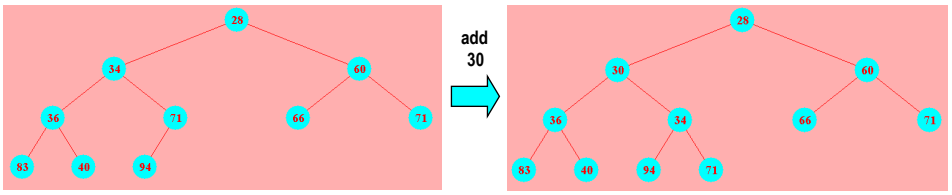


10

Inserting into a heap

to insert into a heap

- place new item in next open leaf position
- if new value is smaller than parent, then swap nodes
- continue up toward the root, swapping with parent, until smaller parent found



note: insertion maintains completeness and the heap property

- worst case, if add smallest value, will have to swap all the way up to the root
- but only nodes on the path are swapped → O(height) = O(log N) swaps

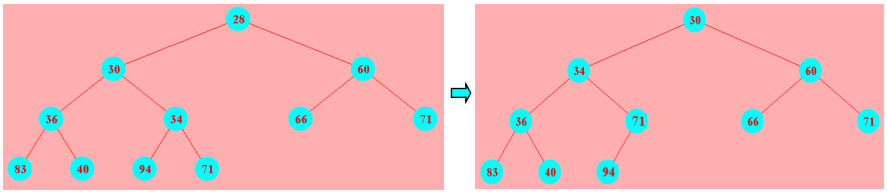
11

Finding/removing from a heap

finding the min value in a heap is O(1) – it's in the root

removing the min value requires some work

- replace root with last node on bottom level
- if new root value is greater than either child, swap with smaller child
- continue down toward the leaves, swapping with smaller child, until smallest



note: removing root maintains completeness and the heap property

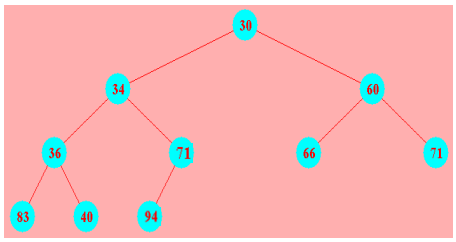
- worst case, if last value is largest, will have to swap all the way down to leaf
- but only nodes on the path are swapped → O(height) = O(log N) swaps

12

Implementing a heap

a heap provides for O(1) find min, O(log N) insertion and min removal

- also has a simple, List-based implementation
- since there are no holes in a heap, can store nodes in an ArrayList, level-by-level



30	34	60	36	71	66	71	83	40	94

- root is at index `0`
- last leaf is at index `size()-1`
- for a node at index `i`, children are at `2*i+1` and `2*i+2`
- to add at next available leaf, simply add at end

Horner's rule

polynomials are used extensively in mathematics and algorithm analysis

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$$

how many multiplications would it take to evaluate this function for some value of x?

W.G. Horner devised a new formula that transforms the problem

$$p(x) = (\ldots (((a_n x + a_{n-1})x + a_{n-2})x + \ldots + a_1)x + a_0$$

can evaluate in only n multiplications and n additions

Problem reduction

in CSC321, we looked at a number of examples of reducing a problem from one form to another

- e.g., generate the powerset (set of all subsets) of an N element set

$S = \{ x_1, x_2, x_3, x_4 \}$ $\quad$ $powerset_S = \{ \{\}, \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_4\} \}$

- PROBLEM REDUCTION: simplify by reducing it to a problem about bit sequences
  can map each subset into a sequence of N bits: $b_i = 1 \rightarrow x_i$ in subset

$\{ x_1, x_4, x_5 \} \leftrightarrow 10011000\ldots0$

much simpler to generate all possible N-bit sequences

{ 0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111,
1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111 }

15

lcm & gcd

consider calculating the least common multiple of two numbers m & n

- BRUTE FORCE: reduce each number to its prime factors
  then multiply (factors in both m & n) (factors only in m) (factors only in n)

  $24 = 2 \bullet 2 \bullet 2 \bullet 3$
  $60 = 2 \bullet 2 \bullet 3 \bullet 5$
  $lcm(24, 60) = (2 \bullet 2 \bullet 3) \bullet (2) \bullet (5) = 12 \bullet 2 \bullet 5 = 120$

- PROBLEM REDUCTION: can recognize a relationship between lcm & gcd

  $lcm(m, n) = m \times n / gcd(m, n)$

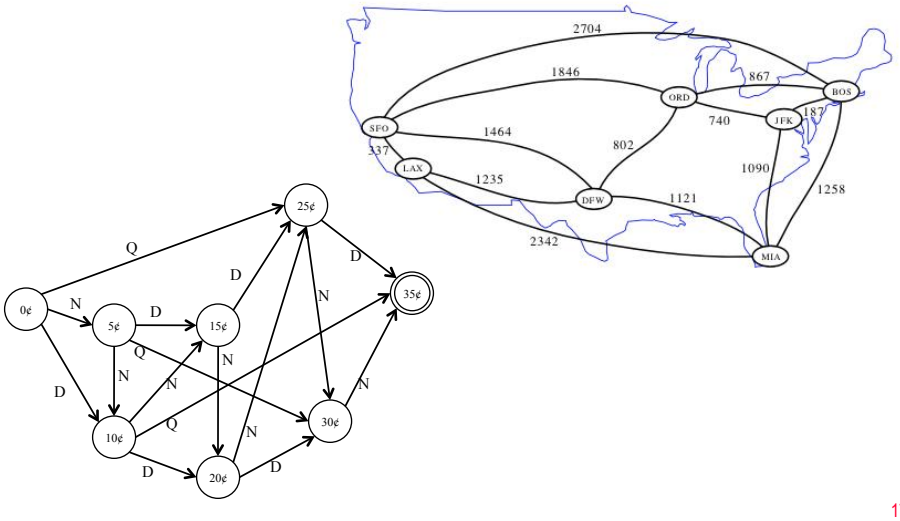
  $lcm(24, 60) = 24 \bullet 60 / 12 = 2 \bullet 60 = 120$

  gcd can be calculated efficiently using Euclid's algorithm:
  $gcd(a, 0) = a$ $\quad$ $gcd(a, b) = gcd(b, a \% b)$

16

Reduction to graph searches

many problems can be transformed into graph problems



Water jug problem

recall from *Die Hard with a Vengeance*

- you have two empty water jugs (4 gallon & 3 gallon capacity) & water supply
- want to end up with exactly 2 gallons in a jug