CSC 421: Algorithm Design & Analysis

Spring 2018

Decrease & conquer

- decrease by constant

  sequential search, insertion sort, topological sort
- decrease by constant factor

  binary search, fake coin problem
- decrease by variable amount

  BST search, selection problem

Decrease & Conquer

based on exploiting the relationship between a solution to a given instance of a problem and a solution to a smaller instance

- once the relationship is established, it can be exploited either top-down or bottom-up

EXAMPLE: sequential search of N-item list

- checks the first item, then searches the remaining sublist of N-1 items

EXAMPLE: binary search of sorted N-item list

- checks the middle item, then searches the appropriate sublist of N/2 items

3 major variations of decrease & conquer

1. decrease by a constant (e.g., sequential search decreases list size by 1)
2. decrease by a constant factor (e.g., binary search decrease list size by factor of 2)
3. decrease by a variable amount

Decrease by a constant

decreasing the problem size by a constant (especially 1) is fairly common

- many iterative algorithms can be viewed this way
  e.g., sequential search, traversing a linked list

- many recursive algorithms also fit the pattern
  e.g., $N! = (N-1)! * N$ $a^N = a^{N-1} * a$

EXAMPLE: insertion sort (decrease-by-constant description)

to sort a list of N comparable items:

1. sort the initial sublist of (N-1) items
2. take the Nth item and insert it into the correct position

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Top-down (recursive) definition

can use recursion to implement this algorithm

- this a *top-down* approach, because it starts with the entire list & recursively works down to the base case

```
public static <T extends Comparable<? super T>>
       void insertionSortRec(T[] items) {
    Decrease.insertionSortRecHelper(items, items.length-1);
}

private static <T extends Comparable<? super T>>
       void insertionSortRecHelper(T[] items, int lastIndex) {
    if (lastIndex > 0) {
        Decrease.insertionSortRecHelper(items, lastIndex-1);
        T value = items[lastIndex];
        int j = lastIndex-1;
        while (j >= 0 && items[j].compareTo(value) > 0) {
            items[j+1] = items[j];
            j--;
        }
        items[j+1] = value;
    }
}
```

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Bottom-up (iterative) definition

for most decrease-by-constant algorithms

- top-down recursion is not necessary and in fact is less efficient
- instead, could use a loop to perform the same tasks in a bottom-up fashion

```
public static <T extends Comparable<? super T>>
       void insertionSort(T[] items) {
    for (int i = 1; i < items.length; i++) {
        T value = items[i];
        int j = i-1;
        while (j >= 0 && items[j].compareTo(value) > 0) {
            items[j+1] = items[j];
            j--;
        }
        items[j+1] = value;
    }
}
```

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Big-Oh of insertion sort

```
public static <T extends Comparable<? super T>>
       void insertionSortRec(T[] items) {
    Decrease.insertionSortRecHelper(items, items.length-1);
}

private static <T extends Comparable<? super T>>
       void insertionSortRecHelper(T[] items, int lastIndex) {
    if (lastIndex > 0) {
        Decrease.insertionSortRecHelper(items, lastIndex-1);
        T value = items[lastIndex];
        int j = lastIndex-1;
        while (j >= 0 && items[j].compareTo(value) > 0) {
            items[j+1] = items[j];
            j--;
        }
        items[j+1] = value;
    }
}
```

analyzing the recursive version

- cost function?
- worst case Big-Oh?

```
public static <T extends Comparable<? super T>>
       void insertionSort(T[] items) {
    for (int i = 1; i < items.length; i++) {
        T value = items[i];
        int j = i-1;
        while (j >= 0 && items[j].compareTo(value) > 0) {
            items[j+1] = items[j];
            j--;
        }
        items[j+1] = value;
    }
}
```

analyzing the iterative version

- worst case Big-Oh?

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Analysis of insertion sort

insertion sort has advantages over other $O(N^2)$ sorts

- best case behavior of insertion sort?
  what is the best case scenario for a sort?
  does insertion sort take advantage of this scenario?

  does selection sort?

- what if a list is partially ordered? (a fairly common occurrence)
  does insertion sort take advantage?

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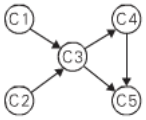
Another example

we considered the problem of *topological sorting* in 321

- given a directed graph, find an ordering of the vertices such that if edge (v_i, v_j) is in the graph, then v_i comes before v_j in the ordering
- as long as there are no cycles in the graph, at least one (and possibly many) topological sorts exists



C1, C2, C3, C4, C5 or C2, C1, C3, C4, C5

topological sorting is useful in many applications

- constructing a job schedule among interrelated tasks
- cell evaluation ordering in spreadsheet formulas
- power rankings of sports teams

- may sometimes generate a topological sort to verify that an ordering is possible, then invest resources into optimizing

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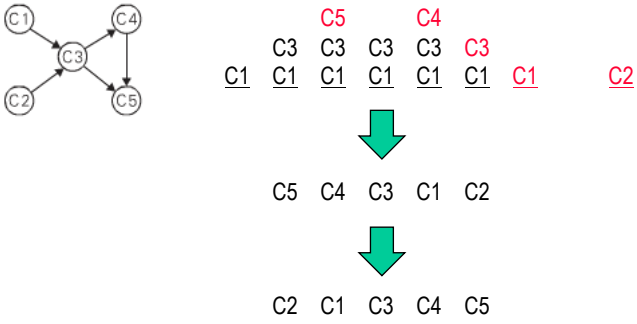
DFS-based topological sort

we considered an algorithm based on depth first search in 321

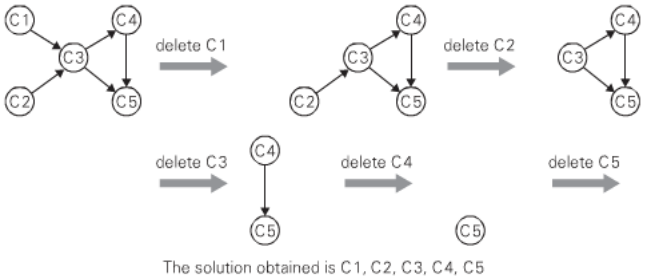
- traverse the graph in a depth-first ordering (using a stack for reached vertices)
- when you reach a dead-end (i.e., pop a vertex off the stack), add it to a list
- finally, reverse the list



Decrease-by-1 topological sort

alternatively,

- while the graph is nonempty
  1. identify a vertex with no incoming edges
  2. add the vertex to a list
  3. remove that vertex and all outgoing edges from it



Decrease by a constant factor

a more efficient (but rare) variation of decrease & conquer occurs when you can decrease by an amount proportional to the problem size

- e.g. binary search → divides a problem of size N into a problem of size N/2

Cost(N)	= Cost(N/2) + C	can unwind Cost(N/2)
	= (Cost(N/4) + C) + C	
	= Cost(N/4) + 2C	can unwind Cost(N/4)
	= (Cost(N/8) + C) + 2C	
	= Cost(N/8) + 3C	can continue unwinding
	= …	(a total of $\log_2 N$ times)
	= Cost(1) + ($\log_2$N)*C	
	= C $\log_2$ N + C'	where C' = Cost(1)
	→ O(log N)	

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Fake coin problem

Given a scale and N coins, one of which is lighter than all the others. Identify the light coin using the minimal number of weighings.

- solution?
- cost function?
- big-Oh?

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What doesn't fit here?

decrease by a constant factor is efficient, but very rare

it is tempting to think of algorithms like merge sort & quick sort

- each divides the problem into subproblems whose size is proportional to the original
- key distinction: these algorithms require solving multiple subproblems, then somehow combining the results to solve the bigger problem
- we have a different name for these types of problems: *divide & conquer*
- NEXT WEEK

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Decrease by a variable amount

sometimes things are not so consistent

- the amount of the decrease may vary at each step, depending on the data

EXAMPLE: searching/inserting in a binary search tree

- if the tree is full & balanced, then each check reduces the current tree into a subtree half the size
- however, if not full & balanced, the sizes of the subtrees can vary
- worst case: the larger subtree is selected at each check
- for a linear tree, this leads to O(N) searches/insertions
- in general, the worst case on each check is selecting the larger subtree
- recall: randomly added values produce enough balance to yield O(log N)

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Selection problem

suppose we want to determine the *kth order statistic* of a list

- i.e., find the kth largest element in the list of N items
  (special case, finding the *median*, is the N/2th order statistic)

- obviously, could sort the list then access the kth item directly
  →O(N log N)

- could do k passes of selection sort (find 1st, 2nd, …, kth smallest items)
  - O(k * N)

- or, we could utilize an algorithm similar to quick sort called *quick select*

  recall, quick sort works by:

  1. partitioning the list around a particular element (i.e., moving all items ≤ the pivot element to its left, all items > the pivot element to its right)
  2. recursively quick sorting each partition

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Lomuto partition

we will assume that the first item is always chosen as pivot value

- simple, but "better" strategies exist for choosing the pivot

- mark the first index as the pivot index
- traverse the list, for each item < the pivot value
  increment the pivot index & swap the item into that index
- finally, swap the pivot value into the pivot index

5	8	7	2	1	6	4
5	**8**	7	2	1	6	4
5	8	**7**	2	1	6	4
5	8	7	**2**	1	6	4
5	2	7	8	**1**	6	4
5	2	1	8	7	**6**	4
5	2	1	8	7	6	**4**
5	2	1	4	7	6	8
4	2	1	5	7	6	8

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Lomuto partition implementation

for both quick sort & quick select, we need to be able to partition a section of the list

- assume parameters left & right are the lower & upper indexes of the section to be partitioned

```
private static <T extends Comparable<? super T>>
       int partition(T[] items, int left, int right) {
    T pivot = items[left];
    int pivotIndex = left;
    for (int i = left+1; i <= right; i++) {
        if (items[i].compareTo(pivot) < 0) {
            pivotIndex++;
            T temp = items[pivotIndex];
            items[pivotIndex] = items[i];
            items[i] = temp;
        }
    }
    T temp = items[pivotIndex];
    items[pivotIndex] = items[left];
    items[left] = temp;
    return pivotIndex;
}
```

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Quick select algorithm

to find the kth value in a list:

1. partition the list (using Lomuto's partitioning algorithm)
2. let pivotIndex denote the index that separates the two partitions
3. if k = pivotIndex+1, then
   return the value at pivotIndex
4. if k < pivotIndex+1, then
   recursively search for the kth value in the left partition
5. if k > pivotIndex+1, then
   recursively search for the (k – (pivotIndex+1))th value in the right partition

4	2	1	5	7	6	8

here, pivotIndex = 3

if k = 4, then answer is in index 3 = 5

if k = 2, then find 2nd value in

4	2	1

if k = 5, then find 1st value in

7	6	8

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Quick select implementation

can be implemented recursively or iteratively

- recursive solution requires a private helper method that utilizes the left & right boundaries of the list section
- as with partition, the code must adjust for the changing boundaries

```
public static <T extends Comparable<? super T>>
       T quickSelect(T[] items, int k) {
    return Decrease.quickSelect(items, 0, items.length-1, k);
}

private static <T extends Comparable<? super T>>
       T quickSelect(T[] items, int left, int right, int k) {
    int pivotIndex = Decrease.partition(items, left, right);
    if (k == pivotIndex+1-left) {
        return items[pivotIndex];
    }
    else if (k < pivotIndex+1-left) {
        return Decrease.quickSelect(items, left, pivotIndex-1, k);
    }
    else {
        return Decrease.quickSelect(items, pivotIndex+1,
                                    right, k-(pivotIndex-left+1));
    }
}
```

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Efficiency of quick select

analysis is similar to quick sort

note that partitioning is O(N)

- if the partition is perfectly balanced:

Cost(N)	= Cost(N/2) + $C_1N + C_2$	can unwind Cost(N/2)
	= (Cost(N/4) + $C_1N/2 + C_2$) + $C_1N + C_2$	
	= Cost(N/4) + $3/2C_1N + 2C_2$)	can unwind Cost(N/4)
	= (Cost(N/8) + $C_1N/4 + C_2$) + $3/2C_1N + 2C_2$	
	= Cost(N/8) + $7/4C_1N + 3C_2$	can continue unwinding
	= …	(a total of $\log_2 N$ times)
	≤ Cost(1) + $2C_1N + (\log_2N)C_2$	note: 1 + 1/2 + 1/4 + 1/8 … → 2
	= $2C_1 N + C_2 \log_2N + C'$	where C' = Cost(1)
	→ O(N)	

- in the worst case, the pivot chosen is the smallest or largest value
  this reduces the algorithm to decrease-by-1 → $O(N^2)$
- as long as the partitioning is reasonably balanced, get O(N) behavior in practice
  there is a complex algorithm for choosing the pivot that *guarantees* linear performance

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