CSC 427: Data Structures and Algorithm Analysis

Fall 2011

Divide & conquer (part 2)

- binary trees
  - standard methods: add, contains, remove, size
  - other methods: numOccur, isLeaf, height, …
- binary search trees
  - BST property
  - overload binary tree methods: add, contains
  - search efficiency, balance

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Dividing & conquering trees

since trees are recursive structures, most tree traversal and manipulation operations can be classified as *divide & conquer algorithms*

- can divide a tree into root + left subtree + right subtree
- most tree operations handle the root as a special case, then recursively process the subtrees

- e.g., to display all the values in a (nonempty) binary tree, divide into
  1. *displaying the root*
  2. *(recursively) displaying all the values in the left subtree*
  3. *(recursively) displaying all the values in the right subtree*

- e.g., to count number of nodes in a (nonempty) binary tree, divide into
  1. *(recursively) counting the nodes in the left subtree*
  2. *(recursively) counting the nodes in the right subtree*
  3. *adding the two counts + 1 for the root*

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BinaryTree class

```
public class BinaryTree<E> {
    protected TreeNode root;

    public BinaryTree() {
        this.root = null;
    }

    public void add(E value) { … }

    public boolean remove(E value) { … }

    public boolean contains(E value) { … }

    public int size() { … }

    public String toString() { … }
}
```

to implement a binary tree, need to store the root node

- the root field is "protected" instead of "private" to allow for inheritance
- the empty tree has a null root
- then, must implement methods for basic operations on the collection

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`size` method

divide-and-conquer approach:

BASE CASE: if the tree is empty, number of nodes is 0

RECURSIVE: otherwise, number of nodes is
(# nodes in left subtree) + (# nodes in right subtree) + 1 for the root

note: a recursive implementation requires passing the root as parameter

- will have a public "front" method, which calls the recursive "worker" method

```
public int size() {
    return this.size(this.root);
}

private int size(TreeNode<E> current) {
    if (current == null) {
        return 0;
    }
    else {
        return this.size(current.getLeft()) +
               this.size(current.getRight()) + 1;
    }
}
```

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`contains` method

divide-and-conquer approach:

BASE CASE: if the tree is empty, the item is not found
BASE CASE: otherwise, if the item is at the root, then found

RECURSIVE: otherwise, search the left and then right subtrees

```
public boolean contains(E value) {
    return this.contains(this.root, value);
}

private boolean contains(TreeNode<E> current, E value) {
    if (current == null) {
        return false;
    }
    else {
        return value.equals(current.getData()) ||
               this.contains(current.getLeft(), value) ||
               this.contains(current.getRight(), value);
    }
}
```

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`toString` method

must traverse the entire tree and build a string of the items

- there are numerous patterns that can be used, e.g., in-order traversal

BASE CASE: if the tree is empty, then nothing to traverse

RECURSIVE: recursively traverse the left subtree, then access the root, then recursively traverse the right subtree

```
public String toString() {
    if (this.root == null) {
        return "[]";
    }
    String recStr = this.toString(this.root);
    return "[" + recStr.substring(0,recStr.length()-1) + "]";
}

private String toString(TreeNode<E> current) {
    if (current ==  null) {
        return "";
    }
    return this.toString(current.getLeft()) +
           current.getData().toString() + "," +
           this.toString(current.getRight());
}
```

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Alternative traversal algorithms

pre-order traversal:

BASE CASE: if the tree is empty, then nothing to traverse

RECURSIVE: access root, recursively traverse left subtree, then right subtree

```
private String toString(TreeNode<E> current) {
    if (current ==  null) {
        return "";
    }
    return current.getData().toString() + "," +
           this.toString(current.getLeft()) +
           this.toString(current.getRight());
}
```

post-order traversal:

BASE CASE: if the tree is empty, then nothing to traverse

RECURSIVE: recursively traverse left subtree, then right subtree, then root

```
private String toString(TreeNode<E> current) {
    if (current ==  null) {
        return "";
    }
    return this.toString(current.getLeft()) +
           this.toString(current.getRight()) +
           current.getData().toString() + ",";
}
```

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Exercises

```
/** @return the number of times value occurs in the tree with specified root */
public int numOccur(TreeNode<E> root, E value) {



}
```

```
/** @return the sum of all the values stored in the tree with specified root */
public int sum(TreeNode<Integer> root) {



}
```

```
/** @return the maximum value in the tree with specified root */
public int max(TreeNode<Integer> root) {


}
```

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`add` method

how do you add to a binary tree?

- ideally would like to maintain balance, so (recursively) add to smaller subtree
- big Oh?
- we will more consider efficient approaches for maintaining balance later

```
public void add(E value) {
    this.root = this.add(this.root, value);
}

private TreeNode<E> add(TreeNode<E> current, E value) {
    if (current == null) {
        current = new TreeNode<E>(value, null, null);
    }
    else if (this.size(current.getLeft()) <= this.size(current.getRight())) {
        current.setLeft(this.add(current.getLeft(), value));
    }
    else {
        current.setRight(this.add(current.getRight(), value));
    }
    return current;
}
```

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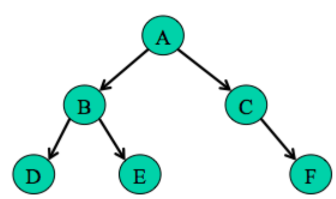
`remove` method

how do you remove from a binary tree?

- tricky, since removing an internal node means rerouting pointers
- must maintain binary tree structure



simpler solution

1. find node (as in search)
2. if a leaf, simply remove it
3. if no left subtree, reroute parent pointer to right subtree
4. otherwise, replace current value with a leaf value from the left subtree (and remove the leaf node)

DOES THIS MAINTAIN BALANCE?
(you can see the implementation in BinaryTree.java)

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HW4: more BinaryTree methods

you are to implement and test the following:

- isLeaf: whether an item appears in the tree in a leaf node
- isParent: whether an item appears in the tree in a non-leaf node
- numLeaves: the number of leaf nodes in the tree
- numParents: the number of non-leaf nodes in the tree
- height: the height of the tree (i.e., length of longest path)
- weight: the weight of the tree (i.e., sums of depths of each node)

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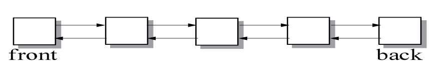
Searching linked lists

recall: a (linear) linked list only provides sequential access → O(N) searches



it is possible to obtain O(log N) searches using a tree structure

in order to perform binary search efficiently, must be able to

- access the middle element of the list in O(1)
- divide the list into halves in O(1) and recurse

HOW CAN WE GET THIS FUNCTIONALITY FROM A TREE?

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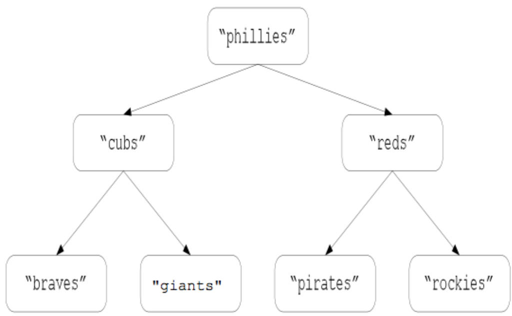
Binary search trees

a *binary search tree* is a binary tree in which, for every node:

- the item stored at the node is ≥ all items stored in its left subtree
- the item stored at the node is < all items stored in its right subtree



in a (balanced) binary search tree:

- middle element = root
- 1st half of list = left subtree
- 2nd half of list = right subtree

furthermore, these properties hold for each subtree

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BinarySearchTree class

can use inheritance to derive BinarySearchTree from BinaryTree

```
public class BinarySearchTree<E extends Comparable<? super E>>
extends BinaryTree<E> {

    public BinarySearchTree() {
        super();
    }

    public void add(E value) {
        // OVERRIDE TO MAINTAIN BINARY SEARCH TREE PROPERTY
    }

    public void CONTAINS(E value) {
        // OVERRIDE TO TAKE ADVANTAGE OF BINARY SEARCH TREE PROPERTY
    }

    public void remove(E value) {
        // DOES THIS NEED TO BE OVERRIDDEN?
    }
}
```

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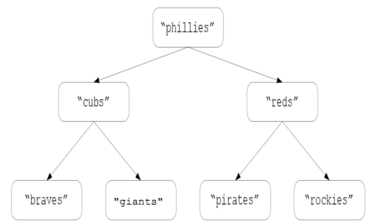
Binary search in BSTs



to search a binary search tree:

1. if the tree is empty, NOT FOUND
2. if desired item is at root, FOUND
3. if desired item < item at root, then recursively search the left subtree
4. if desired item > item at root, then recursively search the right subtree

```
public boolean contains(E value) {
    return this.contains(this.root, value);
}

private boolean contains(TreeNode<E> current, E value) {
    if (current == null) {
        return false;
    }
    else if (value.equals(current.getData())) {
        return true;
    }
    else if (value.compareTo(current.getData()) < 0) {
        return this.contains(current.getLeft(), value);
    }
    else {
        return this.contains(current.getRight(), value);
    }
}
```

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Search efficiency

how efficient is search on a BST?

- in the best case?

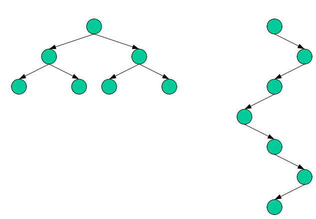
  O(1) if desired item is at the root

- in the worst case?

  O(height of the tree) if item is leaf on the longest path from the root

in order to optimize worst-case behavior, want a (relatively) balanced tree

- otherwise, don't get binary reduction
- e.g., consider two trees, each with 7 nodes



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How deep is a balanced tree?

THEOREM: A binary tree with height H can store up to 2^H-1 nodes.

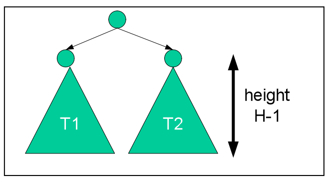
Proof (by induction):

BASE CASES: when H = 0, $2^0 - 1 = 0$ nodes ✓

when H = 1, $2^1 - 1 = 1$ node ✓

HYPOTHESIS: assume a tree with height H-1 can store up to $2^{H-1} -1$ nodes

INDUCTIVE STEP: a tree with height H has a root and subtrees with height up to H-1



by our hypothesis, T1 and T2 can each store $2^{H-1} -1$ nodes, so tree with height H can store up to

$1 + (2^{H-1} -1) + (2^{H-1} -1) =$

$2^{H-1} + 2^{H-1} -1 =$

$2^H -1$ nodes ✓

equivalently: N nodes can be stored in a binary tree of height $\lceil \log_2(N+1) \rceil$

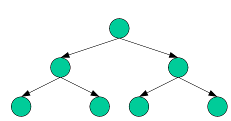
Search efficiency (cont.)

so, in a balanced binary search tree, searching is O(log N)

N nodes → height of ⌈log2(N+1)⌉ → in worst case, have to traverse ⌈log2(N+1)⌉ nodes

what about the average-case efficiency of searching a binary search tree?

- assume that a search for each item in the tree is equally likely
- take the cost of searching for each item and average those costs



costs of search

1

2 + 2

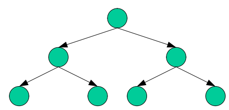
3 + 3 + 3 + 3

→ 17/7 → 2.42

define the *weight* of a tree to be the sum of all node depths (root = 1, …)

average cost of searching a tree = weight of tree / number of nodes in tree

Search efficiency (cont.)



costs of search

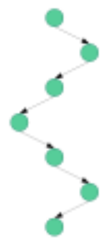
1

2 + 2

3 + 3 + 3 + 3

→ 17/7 → 2.42

~log N



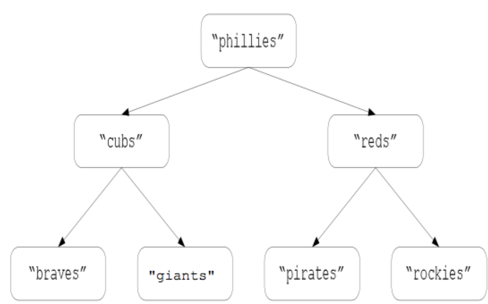
costs of search

1
+ 2
+ 3
+ 4
+ 5
+ 6
+ 7

→ 28/7 → 4.00

~N/2

Inserting an item



inserting into a BST

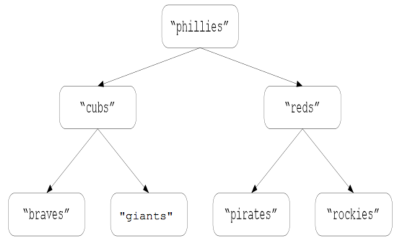
1. traverse edges as in a search
2. when you reach a leaf, add the new node below it

```
 public void add(E value) {
    this.root = this.add(this.root, value);
}

private TreeNode<E> add(TreeNode<E> current, E value) {
    if (current == null) {
        return new TreeNode<E>(value, null, null);
    }

    if (value.compareTo(current.getData()) <= 0) {
        current.setLeft(this.add(current.getLeft(), value));
    }
    else {
        current.setRight(this.add(current.getRight(), value));
    }
    return current;
}
```

Removing an item



recall BinaryTree remove

1. find node (as in search)
2. if a leaf, simply remove it
3. if no left subtree, reroute parent pointer to right subtree
4. otherwise, replace current value with a leaf value from the left subtree (and remove the leaf node)

CLAIM: as long as you select the rightmost (i.e., maximum) value in the left subtree, this remove algorithm maintains the BST property

WHY?

so, no need to override remove

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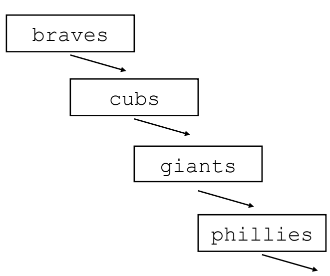
Maintaining balance

PROBLEM: random insertions (and removals) do not guarantee balance

- e.g., suppose you started with an empty tree & added words in alphabetical order
  braves, cubs, giants, phillies, pirates, reds, rockies, …



with repeated insertions/removals, can degenerate so that height is O(N)

- specialized algorithms exist to maintain balance & ensure O(log N) height
- or take your chances

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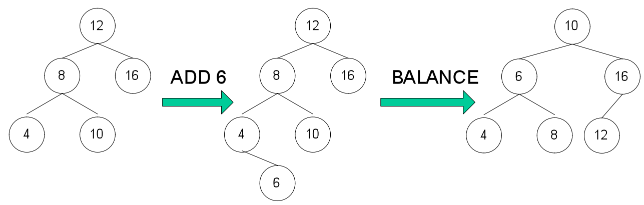
Balancing trees

on average, N random insertions into a BST yields O(log N) height

- however, degenerative cases exist (e.g., if data is close to ordered)

we can ensure logarithmic depth by maintaining balance



maintaining full balance can be costly

- however, full balance is not needed to ensure O(log N) operations (LATER)

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